turbulence to approach the freestream value at a faster rate when compared to the smooth, solid wall.

The normal turbulence intensity results are shown in Fig. 3. The normal turbulence was observed to have a higher value near the wall in comparison to a smooth, solid wall. The normal turbulence also approached the freestream value faster with increasing injection velocity.

The Reynolds stress profiles are presented in Fig. 4. The no injection case for the perforated titanium wall shows a larger Reynolds stress near the wall than does the smooth, solid wall. Injection causes a much larger Reynolds stress, and this is seen to increase with increasing injection velocity. The Reynolds stress for the perforated titanium wall is observed to approach the freestream value in a manner similar to that of the axial and normal turbulence; however, this behavior is not as pronounced.

The local skin friction coefficients and corresponding wall shear data for the perforated titanium wall with and without injection along with previous results for a sintered metal surface in the same apparatus are shown in Fig. 5. Comparison of the results for the no injection case show that the perforated titanium wall has about a 30% higher skin friction coefficient than does the smooth, solid wall. The porous, sintered wall was observed to have a 43% higher skin friction coefficient. The results show that for a given V_0^+, the sintered material is more effective in the reduction of skin friction coefficient. A reduction of 33% in C_f is achieved at $V_0^+=0.10$ for the sintered material relative to the no blowing case. In comparison, only a 28% reduction in skin friction coefficient is achieved at higher $V_0^+=0.19$ for the perforated titanium wall. Since the porous, sintered wall is more uniformly porous on the small scale, these results are understandable. The perforated wall produced an array of tiny air jets. Lastly, considerable blowing was required to reduce the C_f of both porous walls below the value of the smooth, solid wall.

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Momentum/Heat-Transfer Analogy for Turbulent Boundary Layers in Mild Pressure Gradients

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Introduction

THE heat-transfer rate can be estimated from the skin friction using momentum/heat-transfer analogies without actually solving the energy balance equation. The analogies have been widely adopted for the evaluation of local heat (mass)-transfer rates in the turbulent boundary layers developed even on rotating surfaces.[1,2]

Various analysts[3-5] proposed different formulas on the basis of the Couette flow approximation. However, Spalding[6] employed the von Mises transformation to retain the advection terms. None of these analyses takes account of the pressure gradient effects.

There are only a few methods proposed for nonzero pressure gradients. Reshotko and Tucker[7] developed the method for nonzero pressure gradients in which the Reynolds analogy factor depends on the velocity shape factor alone. Cohen[8] used the integral momentum and energy equations with a number of approximations and obtained the Reynolds analogy factor for compressible flows under nonzero pressure gradients, but the analogy factor is found to become independent of the pressure gradients as the Mach number approaches to zero. Tetervin[9] approximated the distributions of the shear stress and heat flux by third-order polynomials and computed the Reynolds analogy factor for the Prandtl number unity as a function of the velocity shape factor and a Clauser-type pressure gradient parameter. He simply multiplied the resulting analogy factor by Colburn's Reynolds analogy factor so as to take account of the Prandtl number effects. This treatment for the Prandtl number effects seems to lack a sound basis. Because of the limitations of the methods cited above, a different approach seems desirable for the evaluation of the Reynolds analogy factor.

This Note introduces the velocity law of the wall valid for mild pressure gradients and couples it to the temperature law of the wall to obtain the analogy factor as influenced by the pressure gradients and laminar Prandtl numbers (the laminar Prandtl number is varied while the turbulent Prandtl number is assumed to be unity). It will be shown that the resulting general formula for the momentum/heat-transfer analogy is valid for all laminar Prandtl numbers except those characteristic of liquid metals.

Analysis

The mixing length hypothesis under the linear scale variation leads to

$$\tau=\rho\left(\kappa y\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \tag{1}$$

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where u and τ are, respectively, the mean velocity in the streamwise direction x and the local shear stress at the normal distance y away from the wall. The density and von Kármán constant are denoted by ρ and κ, respectively. The above equation can be obtained also from the turbulent kinetic energy balance equation under the structural similarity assumption with negligible diffusion rate. The shear stress near the wall is usually found to vary linearly as

$$\tau \cong \tau_w + \left(\frac{d\tau}{dy}\right)_w y \tag{2}$$

where the subscript w refers to the wall. Equations (1) and (2) can be combined to give

$$\frac{du}{dy} = \frac{(\tau/\rho)^{1/2}}{\kappa y} = \left[1+\beta_\delta\left(\frac{y}{\delta}\right)\right]^{1/2}\frac{(\tau_w/\rho)^{1/2}}{\kappa y}$$

$$= \left[1+\sum_1^\infty \binom{1/2}{n}\left(\frac{\beta_\delta y}{\delta}\right)^n\right]\frac{(\tau_w/\rho)^{1/2}}{\kappa y} \tag{3a}$$

where

$$\beta_\delta \equiv \frac{\delta}{\tau_w}\left(\frac{d\tau}{dy}\right)_w \tag{3b}$$

$$\binom{1/2}{n} = 1/2\,(1/2-1)\,(1/2-2)\ldots.(1/2-n+1)/n! \tag{3c}$$

and δ is the viscous (velocity) boundary-layer thickness. The stress gradient is assumed to be mild enough that $|\beta_\delta y/\delta| < 1$. The reason why τ is expanded in this way (even when a closed form for u may easily be obtained from its original differential form) will shortly be apparent. Equation (3a) may readily be integrated to yield

$$\frac{u}{(\tau_w/\rho)^{1/2}} = \frac{1}{\kappa}\ln\left(\frac{y}{y_s}\right) + \frac{1}{\kappa}\sum_1^\infty \binom{1/2}{n}\frac{\beta_\delta^n}{n}\left(\frac{y^n - y_s^n}{\delta^n}\right) \tag{4}$$

The comparison of Eq. (4) with the conventional "law of the wall" ($\beta_\delta = 0$) implies

$$y_s(\tau_w/\rho)^{1/2}/\nu \cong e^{-\kappa B} \cong 0.1 \tag{5}$$

where the von Kármán constant κ and the wall law intercept B are assumed to be 0.4 and 5.5, respectively.

Since the advection terms become small near the wall, the momentum and energy equations reduce to

$$\frac{d\tau}{dy} \cong \frac{dp}{dx} \quad \text{and} \quad \frac{dq}{dy} \cong 0 \tag{6a,b}$$

where the pressure and heat flux are denoted by p and q. Equations (6) imply that the temperature profile near the wall may become fairly insensitive to the pressure gradient, while the velocity profile there must correspond to the pressure gradient according to Eqs. (6a) and (3b),

$$\beta_\delta \cong \frac{\delta}{\tau_w}\frac{dp}{dx} \cong -\frac{\rho\delta}{\tau_w}u_e\frac{du_e}{dx} \tag{7}$$

The preceding observation on the energy equation indicates that the temperature law of the wall for zero pressure gradient given below may well be valid even for the case of mild pressure gradients:

$$\frac{\rho Cp(\tau_w/\rho)^{1/2}}{q_w}(T_W - T) = \frac{1}{\kappa}\ln\left(\frac{y}{y_s}\right) + P(Pr) \tag{8}$$

where T and Cp are the temperature and specific heat, respectively, and $P(Pr)$ the Jayatillaka "P-function[10]" that accounts for the enhanced resistance to heat transfer offered by the viscous sublayer as a function of laminar Prandtl number Pr,

$$P(Pr) = 9.24(Pr^{3/4} - 1) \tag{9}$$

In Eqs. (8)) and (9), the turbulent Prandtl number is assumed to be unity. It should be noted that the temperature law of the wall [Eq. (8)] becomes invalid for fluids with very small Prandtl numbers, such as liquid metals. After evaluating Eqs. (4) and (8) at the viscous ($y=\delta$) and the thermal ($y=\delta_t$) boundary-layer edges, respectively, the subtraction of Eq. (8) from Eq. (4) leaves

$$\left(\frac{2}{C_{fx}}\right)^{1/2} - \frac{(C_{fx}/2)^{1/2}}{S_{tx}} = \frac{1}{\kappa}\ln\left(\frac{\delta}{\delta_t}\right) - P(Pr)$$

$$+\frac{1}{\kappa}\sum_1^\infty \binom{1/2}{n}\frac{\beta_\delta^n}{n}\left[1-\left(\frac{y_s}{\delta}\right)^n\right] \tag{10a}$$

where the skin friction coefficient is

$$C_{fx} \equiv 2\tau_w/\rho u_e^2 \tag{10b}$$

and the Stanton number

$$S_{tx} \equiv q_w/\rho Cp u_e(T_w - T_e) \tag{10c}$$

The subscript e refers to the corresponding boundary-layer edge $y=\delta$ or δ_t. Due to Eq. (5), (y_s/δ) in the last term on the right-hand side of Eq. (10a) may be dropped. Moreover, the logarithmic term in Eq. (10a) can be neglected since $\ln(\delta/\delta_t) \sim 0$ for $Pr \sim 1$ and $\ln(\delta/\delta_t) \ll \kappa P(Pr)$ for $Pr \gg 1$. Thus, Eq. (10a) reduces to the following compact form for the momentum/heat-transfer analogy of present concern:

$$\frac{2S_{tx}}{C_{fx}} \cong \left\{1+\left(\frac{C_{fx}}{2}\right)^{1/2}\left[P(Pr) - \frac{1}{\kappa}\sum_1^\infty \binom{1/2}{n}\frac{\beta_\delta^n}{n}\right]\right\}^{-1} \tag{11}$$

Equation (11) immediately provides S_{tx} for given C_{fx}, β_δ, and Pr.

Results

Although a number of elaborate calculation schemes for the turbulent boundary layers[11-13] are now available, a simple momentum integral approach would suffice for rough estimates of C_{fx} and β_δ so that the validity of Eq. (11) may be substantiated. A usual control volume analysis leads to the momentum balance relation as given by

$$\frac{d}{dx}\int_0^\delta (u_e u - u^2)\,dy + \frac{du_e}{dx}\int_0^\delta (u_e - u)\,dy = \frac{\tau_w}{\rho} \tag{12}$$

A Blasius-type expression may be sufficient for the evaluation of the local skin friction coefficient

$$C_{fx} = 0.045(\nu/u_e\delta)^{1/4} \tag{13}$$

which corresponds to the one-seventh power law as

$$u/u_e = (y/\delta)^{1/7} \tag{14}$$

Upon substitution of Eqs. (13) and (14), Eq. (12) may easily be solved for δ. The solution is given by

$$(\delta/x)Rex^{1/5} = 0.371\, I^{4/5} \tag{15a}$$

where

$$I=\int_0^x u_e^{27/7}\mathrm{d}x/u_e^{27/7}x \tag{15b}$$

and

$$Rex=u_e x/\nu \tag{15c}$$

The function I obviously becomes unity for the case of a flat plate at zero incidence. The substitution of Eq. (15a) into Eqs. (13) and (7) yields

$$C_{fx}Rex^{1/5}=0.0577\ /\ I^{1/5} \tag{16a}$$

and

$$\beta_\delta=-\frac{2m\delta}{xC_{fx}}=-\frac{90}{7}mI \tag{16b}$$

where

$$m\equiv\frac{\mathrm{d}\ell n u_e}{\mathrm{d}\ell n x} \tag{16c}$$

For the special case of constant m, Eq. (16c) leads to the wedge flow for which

$$u_e \propto x^m \tag{17a}$$

$$I=[1+(27/7)m]^{-1} \tag{17b}$$

The Reynolds analogy factor for a flat plate ($m=0$) obtained on the basis of Eq. (11) with an aid of Eq. (16a) is indicated in the Fig. 1 along with the Colburn, Deissler,[4] and Spalding[6] analogies for laminar Prandtl numbers ranging from those characteristic of gases ($Pr\sim1$) to those of oils and for $Rex=10^7$. The curves of Deissler and Spalding are taken directly from Ref. 14. The level of the present analysis appears to be in good accord with the numerical integration results of Deissler and Spalding. The agreement of the present analogy with that of Deissler is especially good, even though they have been obtained in significantly different manners. All three analogies approach an asymptotic limit proportional to $Pr^{-3/4}$ for large Prandtl numbers (instead of $Pr^{-2/3}$ as in the Colburn analogy). The Colburn analogy, which may be regarded as a direct extension of the laminar thermal boundary-layer theory, yields erroneous results for large Prandtl numbers.

Since $C_{fx}^{1/2}$ is proportional to $Rex^{-1/10}$ according to Eq. (16a), the coefficients associated with β_δ^n in Eq. (11) are fairly insensitive to Rex. In fact, $(C_{fx}/2)^{1/2}$ varies roughly from 0.05 to 0.03 in the range $5\times10^5<Rex<10^7$ in which the power law is believed to be accurate. Thus, for gases in this Rex range, Eq. (11) may well be approximated by

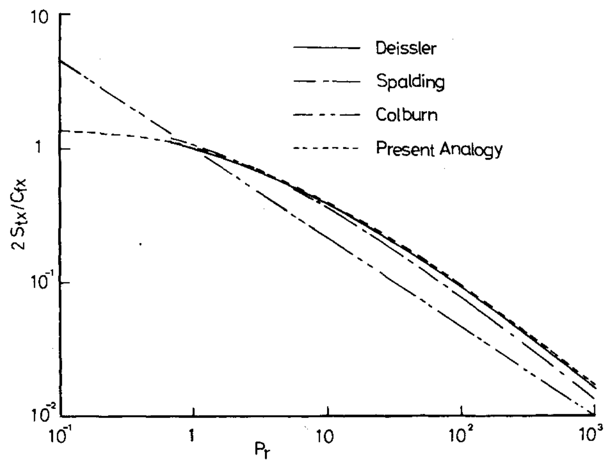

Fig. 1 Reynolds analogy factor, $\beta_\delta=0$ and $Rex=10^7$.

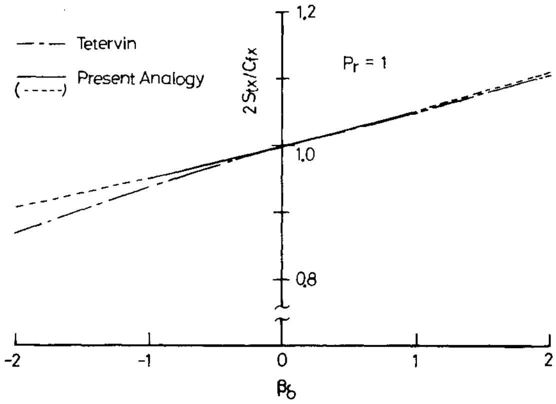

Fig. 2 Effects of pressure gradient, $Pr=1$.

$$\frac{2S_{tx}}{C_{fx}}\cong\left[1-0.1\sum_1^\infty\binom{1/2}{n}\frac{\beta_\delta^n}{n}\right]^{-1} \tag{18}$$

where $(1/\kappa)(C_{fx}/2)^{1/2}$ on the right-hand side of Eq. (11) is estimated to be in the vicinity of 0.1. The series expansion in Eq. (18) converges quickly within the radius of convergence, $|\beta_\delta|<1$. The following first-order approximation turns out to be quite sufficient for the estimation of the Reynolds analogy factor:

$$2S_{tx}/C_{fx}\cong(1-0.05\beta_\delta)^{-1} \tag{19}$$

Equation (19) of the present analogy is plotted in Fig. 2 with the numerical results obtained by Tetervin[9] for the ratio of the displacement thickness to the momentum thickness $1.3\cong9/7$ corresponding to the one-seventh power law. Since the laminar separation ($m=-0.091$) and the plane stagnation flow ($m=1$) correspond to $\beta_\delta\cong1.80$ and -2.65 according to Eqs. (16b) and (17b), β_δ may well be in the range $|\beta_\delta|<2$ for the usual practical applications. Figure 2 shows that both analogy factors are in fair agreement with each other even in this wide range of β_δ, despite the fact that the series expansion would diverge for $|\beta_\delta|>1$.

Concluding Remarks

The attempt to verify the validity of the present formula through a comparison with experimental data had to be abandoned since there are at present very few reliable turbulent data obtained under variable pressure fields. However, the present approach for nonzero pressure gradients is believed to be sound. Careful experiments may be needed if the accuracy of the present method is to be examined.

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Explicit Representations of the Complete Velocity Profile in a Turbulent Boundary Layer

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Nomenclature

B	= constant in the logarithmic law
$e(y^+)$	= difference of the approximation, Eq. (10)
$f(y^+)$	= function representing the law of the wall
g	= function representing the law of the wake
Re_θ	= Reynolds number based on momentum thickness
u	= mean velocity component parallel to the wall
u_e	= mean velocity at the boundary-layer edge
u_τ	= friction velocity, $(\tau_w/\rho)^{1/2}$
u^+	= dimensionless u velocity, u/u_τ
w	= wake function
y	= coordinate normal to the wall
y^+	= dimensionless distance from the wall, yu_τ/ν
δ, δ^*, θ	= boundary-layer, displacement, and momentum thickness, respectively
κ	= von Kármán constant
ν	= molecular kinematic viscosity
Π	= Coles wake parameter
ρ	= fluid density
τ_w	= wall shear stress

Introduction

It is well known that most of the commonly used turbulence models lead to parabolic systems when coupled with the boundary-layer equations. Consequently, starting a differential method of predicting turbulent boundary layers requires the specification of initial profiles for the dependent variables. For this purpose, an accurate and computationally convenient expression for the mean velocity distribution is of particular importance to users of existing computer codes and to developers of new calculation methods. For the problem at hand, a "computationally convenient" formula means a representation of the mean-velocity profile which has some or preferably all of the following characteristics: 1) it is a closed-form expression, 2) it gives u explicitly as a function of y, 3) it is valid over the whole width of the boundary layer, and 4) it is relatively easily evaluated. The objective of this Note is to provide such a formula for external boundary layers and pipe flows. The analysis holds for two-dimensional incompressible turbulent flow past a smooth surface but (being within the framework of the wall-wake similarity laws) fails in the cases of relaxing flows and flows characterized by the presence of very large positive or negative pressure gradients.[1]

Coles[2] has shown that an expression of the form

$$u^+ = \frac{1}{\kappa}\ln y^+ + B + g\left(\Pi, \frac{y}{\delta}\right) \tag{1}$$

provides an accurate fit to experimental velocity data for both equilibrium and nonequilibrium turbulent boundary layers for $y^+ > 50$. However, in order to obtain velocity profiles valid over the whole width of the boundary layer one can write Eq. (1) in a slightly different way

$$u^+ = f(y^+) + g(\Pi, y/\delta) \tag{2}$$

where $f(y^+)$ is a representation of the law of the wall valid over the whole inner layer and is asymptotic at large y^+ to $(1/\kappa)\ln y^+ + B$. Function $g(\Pi, y/\delta)$ is a representation of the law of the wake. In Eqs. (1) and (2), Π is Coles' wake parameter.[1] For the function $g(\Pi, y/\delta)$ we adopt the expression

$$g\left(\Pi, \frac{y}{\delta}\right) = \frac{1}{\kappa}(1+6\Pi)\left(\frac{y}{\delta}\right)^2 - \frac{1}{\kappa}(1+4\Pi)\left(\frac{y}{\delta}\right)^3 \tag{3}$$

proposed independently by Finley et al.,[3] Granville,[4] and Dean.[5] Equation (3) is an improvement over the more widely used form

$$g\left(\Pi, \frac{y}{\delta}\right) = \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right) = \frac{2\Pi}{\kappa}\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \tag{4}$$

since it describes more accurately the true boundary conditions on the wake function.[4] In the present analysis it is assumed that in addition to u_e and ν the parameters u_τ, δ, and Π are known at the streamwise station in question. This is not a stringent requirement since any one of the five parameters u_τ, δ, Π, δ^*, θ can be determined if two of them are known.[1]

Inner Layer (Law of the Wall)

A large variety of analytical representations of the law of the wall have been proposed, characterized by various levels of complexity and accuracy. Spalding's implicit formula may be considered the most widely used and is adopted by White[6] and Dean[5] as the accepted form of the law of the wall.

Spalding[7] has shown that Laufer's experimental data[8] for the mean velocity distribution in the inner layer are well fitted

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